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LETTER TO THE EDITOR

Interference in the radiation of two point-like sources

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Abstract
The interference part of energy–momentum radiated by two point charges arbitrarily moving in flat spacetime is evaluated. It is shown that the sum of work done by Lorentz forces of charges acting on one another exhausts the effect of combination of outgoing electromagnetic waves generated by the charges.

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1. Introduction

In classical electrodynamics particles interact with one another through the medium of a field which has its own uncountable infinite degrees of freedom. The dynamics of the entire system is governed by the action which is invariant under ten infinitesimal transformations (space–time translations and rotations) which constitute the Poincaré group. These symmetry properties imply the conservation laws, i.e. those quantities that do not change with time.

Dirac [1] used the solution of a wave equation with point-like source in the law of conservation of total energy–momentum of a composite (particle plus field) system to derive the radiation-reaction force. In this letter we trace a series of stages in the calculation of the energy–momentum [2]

$$p_{\rm em}^{\nu} = \int_{\Sigma} {\rm d}\sigma_{\mu} T^{\mu\nu}, \tag{1.1}$$

emitted by two relativistic charges. We show that the energy–momentum balance equations result in the law of mutual interaction of these charges.

The evaluation is not a trivial matter, since the Maxwell energy–momentum tensor density $\hat{T}$ is quadratic in electromagnetic field strengths $\hat{f} = \hat{f}_{(1)} + \hat{f}_{(2)}$, where $\hat{f}_{(a)}$ denotes the retarded Liénard–Wiechert solution associated with the ath particle. Hence $\hat{T}$ can be decomposed as follows [3]:

$$T^{\mu\nu} = T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + T^{\mu\nu}_{\rm int}. \tag{1.2}$$

By $\hat{T}_{(a)}$ we mean the contribution due to the field of the ath particle

$$4\pi T^{\mu\nu}_{(a)} = f^{\mu\lambda}_{(a)} f^{\nu}_{(a)\lambda} - 1/4\eta^{\mu\nu} f^{\kappa\lambda}_{(a)} f^{(a)}_{\kappa\lambda} \tag{1.3}$$

while the interference term

$$4\pi T^{\mu\nu}_{\text{int}} = f^{\mu\lambda}_{(1)} f^{\nu}_{(2)\lambda} + f^{\mu\lambda}_{(2)} f^{\nu}_{(1)\lambda} - 1/4\eta^{\mu\nu}\left(f^{\kappa\lambda}_{(1)} f^{(2)}_{\kappa\lambda} + f^{\kappa\lambda}_{(2)} f^{(1)}_{\kappa\lambda}\right) \tag{1.4}$$

describes the *combination* of outgoing electromagnetic waves.

Aguirregabiria and Bel [3] prove the fundamental theorem that the 'mixed' radiation rate does not depend on the shape of the space-like surface which is used to integrate the Maxwell energy–momentum tensor density. For the specific *plane* motion of the charges the perturbation scheme is elaborated within the framework of predictive relativistic mechanics [4, 5]. The lowest approximation gives the well-known expression for the dipole radiation of two point charges whose motion is governed by Coulomb law [6].

A specific iterative procedure is also proposed by Hojman *et al* [7]. The authors study the system of two charges which are close to each other. They introduce the specific small parameter: *relative velocity* of the particles divided on the speed of light. The radiation of the two-body system as a whole is expressed as a function of the relative velocity of relativistic charges.

Covariant separation of the motion of a system of relativistic particles as a whole from its inner motion [8, 9] is fruitful in the study of radiation effects [10]. To describe the radiation of a relativistic N-body system Klepikov [10] defines the *centre* of a system of radiation events which allows us to synchronize the instants at which electromagnetic waves emitted by point charges combine on the points of a very distant sphere. Fourier analysis is applied to calculate the time-angular distribution of energy–momentum radiated by a system of charged particles. The radiation of a bunch of particles moving in a uniform magnetic field is considered in detail.

The only exact solution is obtained by Villarroel and Rivera [11, 12]. The authors calculate the rate of radiation emitted by two uniformly circling point-like charges. Rigorous calculations show that the interference rate of radiation which escapes to infinity is equal to the rate of work done by Lorentz forces of charges acting on one another. It is worth noting that in [13] the balance between the radiation emitted by two identical charges rotating at opposite ends of the diameter of a fixed circle and the work of Lorentz forces has been established numerically.

2. Coordinate system

To integrate the 'mixed' radiation rate

$$p^{\nu}_{\text{int}} = \int_{\Sigma} \mathrm{d}\sigma_{\mu} T^{\mu\nu}_{\text{int}}, \tag{2.1}$$

we use the hyperplane $\Sigma_t = \{y \in \mathbb{M}_4 : y^0 = t\}$ associated with an unmoving inertial observer. The 'laboratory' time t is a single common parameter defined along all the worldlines ζ_a of the system.

Due to delay in disturbances the intersection of spherical wave fronts

$$S_a(z_a(t_a), t - t_a) = \left\{ y \in \mathbb{M}_4 : (y^0 - t_a)^2 = \sum_i \left(y^i - z^i_a(t_a)\right)^2, y^0 = t, t - t_a > 0 \right\}, \tag{2.2}$$

constitutes support of integral (2.1) [3]. The retarded instants t_1 and t_2 label the points $z_1(t_1) \in \zeta_1$ and $z_2(t_2) \in \zeta_2$ at which the past light cone with vertex at observation point $y \in \Sigma_t$

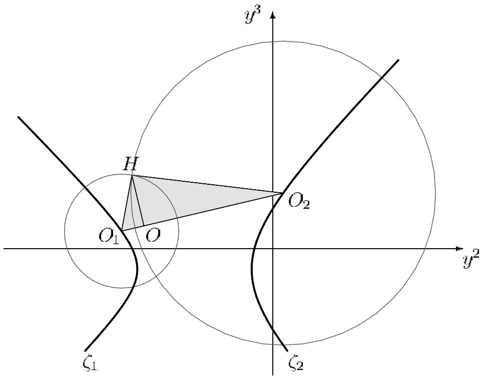

Figure 1. The sphere $S_1(O_1, t - t_1)$ is the intersection of the future light cone with vertex at point $z_1(t_1) \in \zeta_1$ and hyperplane Σ_t. The sphere $S_2(O_2, t - t_2)$ is the intersection of Σ_t and the forward light cone of $z_2(t_2) \in \zeta_2$. Intersection $S_1 \cap S_2$ is the circle $C(O, h)$ with radius $|OH| := h$. It contains an observation point $y \in \Sigma_t$.

is punctured by the worldlines of the first and the second particles, respectively. Points on the circle $S_1 \cap S_2$ are distinguished by polar angle $\varphi \in [0, 2\pi]$ (see figure 1).

Analysis of the triangle $O_1 H O_2$ gives the local expressions for coordinate transformation $(y^\alpha) \mapsto (t, t_1, t_2, \varphi)$:

$$y^\alpha = z_a^\alpha(t_a) + \Omega^\alpha{}_{\alpha'}(t_1, t_2) k_a^{\alpha'}. \tag{2.3}$$

Four components

$$k_a^0 = t - t_a, \qquad k_a^1 = h \sin\varphi, \qquad k_a^2 = h\cos\varphi, \qquad k_a^3 = (-1)^a \frac{q}{2} + \frac{\left(k_2^0\right)^2 - \left(k_1^0\right)^2}{2q} \tag{2.4}$$

constitute null-vector k_a. Having rotated it by orthogonal matrix Ω we obtain the vector K_a pointing from $z_a(t_a) \in \zeta_a$ to $y \in \Sigma_t$. Matrix spacetime components are $\Omega_{0\mu} = \Omega_{\mu 0} = \delta_{\mu 0}$; its space components Ω_{ij} constitute the orthogonal matrix which rotates space axes of the laboratory Lorentz frame until the new z-axis is directed along 3-vector $\mathbf{q} = \mathbf{z}_1(t_1) - \mathbf{z}_2(t_2)$.

An integration hyperplane Σ_t is a surface of constant t. The surface element is given by $\mathrm{d}\sigma_0 = \sqrt{-g}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}\varphi$ where $\sqrt{-g} = r_1 r_2/q$ is the determinant of the metric tensor of Minkowski space viewed in curvilinear coordinates (2.3). Symbol r_a denotes the scalar product $(v_a \cdot K_a)$, $v_a := \left(1, \mathrm{d}z_a^i\big/\mathrm{d}t_a\right)$, taken with opposite sign; r_a is then nothing but the ath retarded distance [2] scaled by factor $\gamma_a^{-1} := \sqrt{1 - \mathbf{v}_a^2}$.

Let us consider the coordinate system centred on an accelerated worldline of the first particle. In context with the principle of retarded causality, Σ_t is divided into two quite different origins: (i) *causal*, which is spanned by curvilinear coordinates (2.3) where t_1 increases from $-\infty$ to the instant $t_1^{\mathrm{ret}}(t)$ being the solution of the algebraic equation

$$t - t_1^{\mathrm{ret}} = q\left(t_1^{\mathrm{ret}}, t\right); \tag{2.5}$$

(ii) *acausal*, where parameter t_1 increases from $t_1^{\mathrm{ret}}(t)$ to the instant of observation t. (The future light cone of $z_1\left(t_1^{\mathrm{ret}}\right)$ touches the second worldline at point $z_2(t)$, see figure 4.) The situation is pictured in figures 2–5.

In an analogous way we construct the coordinate system centred on the worldline of the second particle. If $t_2 \in]-\infty, t_2^{\mathrm{ret}}(t)]$ then $t_1 \in \left[t_1^{\mathrm{ret}}(t_2), t_1^{\mathrm{adv}}(t_2)\right]$; if $t_2 \in \left[t_2^{\mathrm{ret}}(t), t\right]$ then $t_1 \in \left[t_1^{\mathrm{ret}}(t_2), t_1'(t, t_2)\right]$, $\varphi \in [0, 2\pi[$.

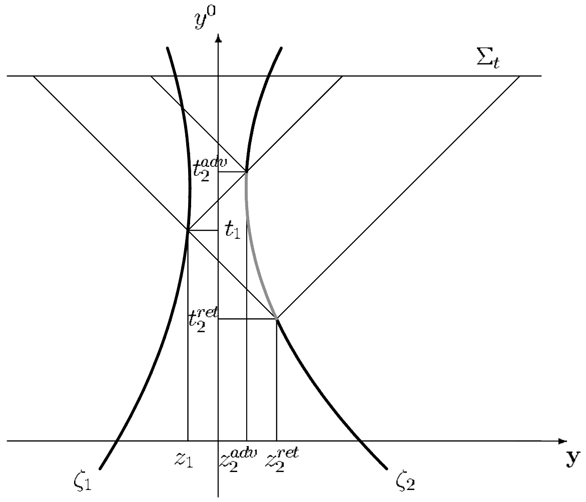

Figure 2. For a given t_1 the retarded time t_2 increases from $t_2^{\mathrm{ret}}(t_1)$ to $t_2^{\mathrm{adv}}(t_1)$. Minimal value $t_2^{\mathrm{ret}}(t_1)$ labels the vertex of the forward light cone which is punctured by the worldline of the first charge at a given point $(t_1, z_1^i(t_1))$. The worldline of the second charge punctures the future light cone of this point at point $(t_2^{\mathrm{adv}}(t_1), z_2^i(t_2^{\mathrm{adv}}))$.

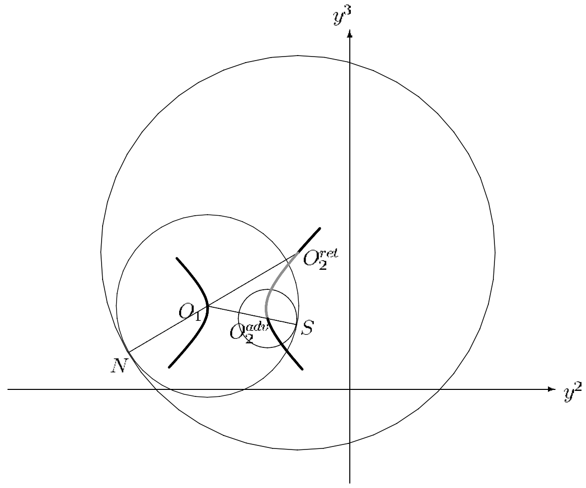

Figure 3. The sphere $S_2(O_2^{\mathrm{ret}}, t - t_2^{\mathrm{ret}})$ is the intersection of the future light cone at $(t_2^{\mathrm{ret}}, z_2^i(t_2^{\mathrm{ret}}))$ and Σ_t. It touches a given sphere $S_1(O_1, t - t_1)$ at point N. The sphere $S_2(O_2^{\mathrm{adv}}, t - t_2^{\mathrm{adv}})$ touches $S_1(O_1, t - t_1)$ at point S. If retarded time t_2 increases from $t_2^{\mathrm{ret}}(t_1)$ to $t_2^{\mathrm{adv}}(t_1)$ the sphere S_1 is covered by circles $C(O, h) = S_1 \cap S_2$. (A circle $S_1 \cap S_2$ is pictured in figure 1.)

3. Interference part of the electromagnetic field 4-momentum

The volume integration (2.1) can be performed via the coordinate system centred on a worldline either of the first particle

$$\left[\int_{-\infty}^{t_1^{\mathrm{ret}}(t)} \mathrm{d}t_1 \int_{t_2^{\mathrm{ret}}(t_1)}^{t_2^{\mathrm{adv}}(t_1)} \mathrm{d}t_2 + \int_{t_1^{\mathrm{ret}}(t)}^{t} \mathrm{d}t_1 \int_{t_2^{\mathrm{ret}}(t_1)}^{t_2'(t,t_1)} \mathrm{d}t_2\right] \int_0^{2\pi} \mathrm{d}\varphi \frac{r_1 r_2}{q} \tag{3.1}$$

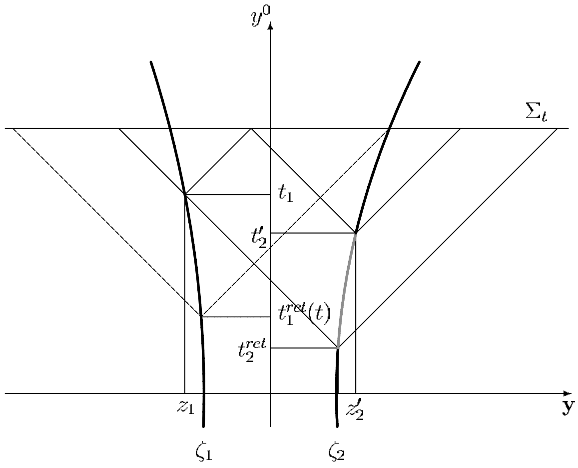

Figure 4. The forward light cone of $(t_1^{\text{ret}}(t), z_1^i(t_1^{\text{ret}}))$ touches the second worldline at the instant of observation. Future light cones of upper vertices do not intersect it at all. For a given $t_1 \in [t_1^{\text{ret}}(t), t]$ the parameter t_2 increases from $t_2^{\text{ret}}(t_1)$ to $t'_2(t, t_1)$. The maximal value $t'_2(t, t_1)$ labels the vertex of the future light cone which touches the forward light cone of $(t_1, z_1^i(t_1))$. The minimal value of t_2 is the solution $t_2^{\text{ret}}(t_1)$ of equation $t_1 - t_2^{\text{ret}} = q(t_1, t_2^{\text{ret}})$.

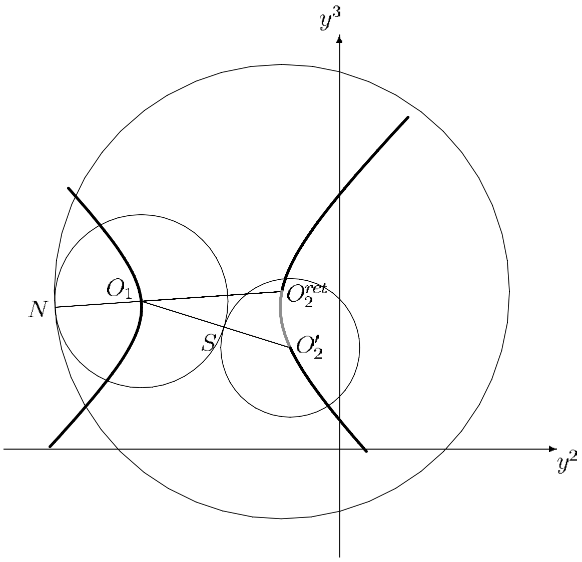

Figure 5. For a given $t_1 \in [t_1^{\text{ret}}(t), t]$ the sphere $S_1(O_1, t - t_1)$ is a disjoint union of circles $C(O, h) = S_1 \cap S_2$. Their radius h and centre coordinate O are determined by t_2. The parameter t_2 increases from $t_2^{\text{ret}}(t_1)$ (circle $S_2(O_2^{\text{ret}}, t - t_2^{\text{ret}})$) to $t'_2(t, t_1)$ (circle $S_2(O'_2, t - t'_2)$); $\varphi \in [0, 2\pi]$.

or of the second particle

$$\left[\int_{-\infty}^{t_2^{\text{ret}}(t)} \mathrm{d}t_2 \int_{t_1^{\text{ret}}(t_2)}^{t_1^{\text{adv}}(t_2)} \mathrm{d}t_1 + \int_{t_2^{\text{ret}}(t)}^{t} \mathrm{d}t_2 \int_{t_1^{\text{ret}}(t_2)}^{t'_1(t,t_2)} \mathrm{d}t_1\right] \int_0^{2\pi} \mathrm{d}\varphi \frac{r_1 r_2}{q}. \tag{3.2}$$

The end points of these integrals arise from the interference pictured in figures 2 and 4.

It is straightforward to substitute the components of electromagnetic fields $\hat{f}_a$ in terms of curvilinear coordinates (t, t_1, t_2, φ) into the integrand of equations (2.1) to calculate the interference part of radiated energy–momentum. First of all we should perform the integration over φ. It is now a straightforward (but tedious) matter to derive that the integral of the interference part of the electromagnetic field's stress–energy tensor over polar angle has the remarkable property of being the sum of partial derivatives in (retarded) times t_1 and t_2 [14]. (The derivation is very cumbersome and we cannot present it in the present letter.)

The crucial issue is that the end points are valuable only in the integration procedure. The retarded instant, $t_a^{\rm ret}(t_b)$, and advanced one, $t_b^{\rm adv}(t_a)$ $(a \neq b)$ arise naturally as the limits of integrals. All the moments are before the observation instant t, so that the retarded causality is not violated. They label the points S and N in which fronts of outgoing electromagnetic waves produced by e_1 and e_2 touch each other (see figures 3 and 5). Triangle O_1O_2H which is pictured in figure 1 reduces to the line at these moments.

It is natural to integrate the expression being the time derivative with respect to t_2 according to the rule (3.1). The result is

$$\left[\int_{-\infty}^{t_1^{\rm ret}(t)} {\rm d}t_1 \int_{t_2^{\rm ret}(t_1)}^{t_2^{\rm adv}(t_1)} {\rm d}t_2 + \int_{t_1^{\rm ret}(t)}^{t} {\rm d}t_1 \int_{t_2^{\rm ret}(t_1)}^{t_2'(t,t_1)} {\rm d}t_2\right] \frac{\partial G_2(t_1,t_2)}{\partial t_2}$$
$$= \int_{-\infty}^{t_1^{\rm ret}(t)} {\rm d}t_1 G_2\big[t_1, t_2^{\rm adv}(t_1)\big] - \int_{-\infty}^{t} {\rm d}t_1 G_2\big[t_1, t_2^{\rm ret}(t_1)\big] + \int_{t_1^{\rm ret}(t)}^{t} {\rm d}t_1 G_2[t_1, t_2'(t,t_1)]. \tag{3.3}$$

Having applied rule (3.2) to the expression of type $\partial G_1/\partial t_1$, we obtain

$$\left[\int_{-\infty}^{t_2^{\rm ret}(t)} {\rm d}t_2 \int_{t_1^{\rm ret}(t_2)}^{t_1^{\rm adv}(t_2)} {\rm d}t_1 + \int_{t_2^{\rm ret}(t)}^{t} {\rm d}t_2 \int_{t_1^{\rm ret}(t_2)}^{t_1'(t,t_2)} {\rm d}t_1\right] \frac{\partial G_1(t_1,t_2)}{\partial t_1}$$
$$= \int_{-\infty}^{t_2^{\rm ret}(t)} {\rm d}t_2 G_1\big[t_1^{\rm adv}(t_2), t_2\big] - \int_{-\infty}^{t} {\rm d}t_2 G_1\big[t_1^{\rm ret}(t_2), t_2\big] + \int_{t_2^{\rm ret}(t)}^{t} {\rm d}t_2 G_1[t_1'(t,t_2), t_2]. \tag{3.4}$$

A smooth double derivative can be written in the form either $\partial/\partial t_1[\partial G_0/\partial t_2]$ or $\partial/\partial t_2[\partial G_0/\partial t_1]$ and coupled with $\partial G_1/\partial t_1$ or $\partial G_2/\partial t_2$, respectively.

An essential feature of integration is that the functions $t_a^{\rm ret}(t_b)$ being the root of algebraic equation

$$t_b - t_a^{\rm ret} = q\big(t_a^{\rm ret}, t_b\big) \tag{3.5}$$

and advanced one, $t_b^{\rm adv}(t_a)$, which satisfies the relation

$$t_b^{\rm adv} - t_a = q\big(t_a, t_b^{\rm adv}\big) \tag{3.6}$$

are inverses to each other. This circumstance allows us to change the variables in the 'advanced' integral in equation (3.3) and then add it to its 'retarded' counterpart in equation (3.4).

The functions $t_1'(t, t_2)$ and $t_2'(t, t_1)$ which satisfy the algebraic equations

$$2t - t_1' - t_2 = q(t_1', t_2), \tag{3.7}$$

$$2t - t_1 - t_2' = q(t_1, t_2'). \tag{3.8}$$

are inverses too. It allows us to change the variables in the last integral of the right-hand side of equation (3.3) and couple it with the last term in equation (3.4).

Scrupulous integration over the 'acausal' region of Σ_t (that pictured in figures 4 and 5) gives the function of the end points only. A surprising feature of the computation is that the result heavily depends on the choice of order of differentiation in double derivatives. For example, if we prefer $\partial/\partial t_1[\partial G_0/\partial t_2]$ in the integrand of the zeroth component of equation (2.1), then we obtain

$$-\frac{e_1e_2}{2k_2^0}\bigg|_{t_2=t_2^{\rm ret}(t)}^{t_2\to t} = -\lim_{t_2\to t}\frac{e_1e_2}{2(t-t_2)} + \frac{e_1e_2}{2\big[t-t_2^{\rm ret}(t)\big]}. \tag{3.9}$$

If one chooses $\partial/\partial t_2[\partial G_0/\partial t_1]$ and adds the term to $\partial G_2/\partial t_2$, we arrive at

$$\frac{e_1e_2}{2k_1^0}\bigg|_{t_2\to t_2^{\rm ret}(t)}^{t_2=t} = \frac{e_1e_2}{2\big[t-t_1^{\rm ret}(t)\big]} - \lim_{t_1\to t}\frac{e_1e_2}{2(t-t_1)}. \tag{3.10}$$

So far as the integration over the causal region of Σ_t is concerned, the result contains, apart from some 'changeable shell', also an 'immovable core' which is then nothing but the sum of work done by the Lorentz forces of interacting particles. The 'shell' consists of the functions of momentary positions and velocities of the particles and involves the divergent terms.

A single charged particle cannot be separated from its bound electromagnetic 'cloud' which has its own 4-momentum and angular momentum [15, 16]. The radiative parts of the Noether conserved quantities lead to an independent existence. The bound terms diverge while the radiative ones are finite. The former depend on the instant characteristics of a charged particle while the latter accumulate with time. And, finally, the form of bound terms crucially depends on the shape of an integration surface while the radiative terms do not depend on Σ.

These circumstances prompt that the 'changeable shell' in a two-body problem is a usual deformation of the bound electromagnetic 'clouds' of charges due to mutual interaction. They are absorbed within the renormalization procedure as well as the inevitable infinities arising in a one-particle problem. The 'immovable' terms should only be taken into account in the total energy–momentum of our composite system:

$$P^\mu = \sum_{a=1}^{2}\left[p_{a,\rm part}^\mu(t) + \frac{2}{3}e_a^2\int_{-\infty}^{t}{\rm d}t_a(a_a\cdot a_a)u_a^\mu(t_a)\right] - \sum_{b\neq a}\int_{-\infty}^{t}{\rm d}t_aF_{ba}^\mu. \tag{3.11}$$

Here $p_{a,\rm part}$ denotes the (already renormalized) 4-momentum of the ath charged particle. The integral of Larmor relativistic rate describes the contributions $\hat{T}_{(a)}$ due to the ath individual field, while the sum of work done by the Lorentz force due to the bth particle acting on the ath one expresses the joint contribution $\hat{T}_{\rm int}$ due to combination of fields.

4. Conclusions

Conserved quantities place stringent requirements on the dynamics of the system. Change in the radiative parts of energy–momentum and angular momentum carried by the electromagnetic field should be balanced by a corresponding change in the (already renormalized) particles' 4-momenta and angular momenta, respectively. Since the action is not propagated instantaneously, the balance in a vicinity of the first charge as well as in a neighbourhood of the second charge should be achieved separately. Having differentiated equation (3.11) we arrive at the relativistic generalization of Newton's second law

$$\dot{p}_{a,\rm part}^\mu = -\frac{2}{3}e_a^2(a_a\cdot a_a)u_a^\mu + F_{ba}^\mu \tag{4.1}$$

where loss of energy due to radiation is taken into account. To obtain the Lorentz–Dirac equation for *more than one charge* we have to substantiate Teitelboim's expression [15] for the 4-momentum of an accelerated point-like charge:

$$p_{a,\rm part}^\mu = mu_a^\mu - \frac{2}{3}e_a^2a_a^\mu. \tag{4.2}$$

The law of conservation of total angular momentum of the system implies (4.2) whenever the interference contribution of the radiative part of angular momentum carried by the electromagnetic field is equal to

$$-\sum_{b\neq a}\int_{-\infty}^{t}\mathrm{d}t_a\left[z_a^\mu(t_a)F_{ba}^\nu - z_a^\nu(t_a)F_{ba}^\mu\right] \tag{4.3}$$

(see [17]). It is true in the case of head-on collision [18]; the problem of electromagnetic angular momentum radiated by two *arbitrarily moving* charges requires careful consideration [19].

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